Algebraic curves
Solution sheet 3

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Unless otherwise specified, k is an algebraically closed field.

Exercise 3.1.

1. Show that $V(Y - X^2) \subset \mathbb{A}^2(\mathbb{C})$ is irreducible; in fact, $I(V(Y - X^2)) = (Y - X^2)$.
2. Decompose $V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3) \subset \mathbb{A}^2(\mathbb{C})$ into irreducible components.
3. Show that $F = Y^2 + X^2(X-1)^2 \in \mathbb{R}[X,Y]$ is an irreducible polynomial, but $V(F)$ is reducible.

Solution 1.

1. I is prime ideal implies that $V(I)$ is irreducible. Now $I = (Y - X^2)$ is prime because $\mathbb{C}[X,Y]/I \simeq \mathbb{C}[X]$ is integral.
2. Let $V := V(Y^4 - X^2, Y^4 - X^2Y^2 + XY^2 - X^3)$. We can see that

$$Y^4 - X^2 = (Y^2 + X)(Y^2 - X)$$

   and

$$Y^4 - X^2Y^2 + XY^2 - X^3 = (Y^2 + X)(Y^2 - X^2)$$

   We see that $Y^2 + X$ is a common irreducible factor so it is an irreducible component of dimension 1 in V. There are two other irreducible components given by points $(1,1)$ and $(1,-1)$ in the intersection of $V(Y^2 - X)$ and $V(Y^2 - X^2) = V((Y - X)(Y + X))$. (Note that $(0,0)$ is already contained in $V(Y^2 + X)$)
3. In $\mathbb{C}[X,Y]$,

$$F = (Y - iX(X-1))(Y + iX(X-1))$$

   By unicity of the decomposition in irreducible factors and as $i \notin \mathbb{R}$, F is irreducible in $\mathbb{R}[X,Y]$. However, $V(F) = V(Y - X(X-1)) \cup V(Y + X(X-1))$

Exercise 3.2.

1. Consider the twisted cubic curve $C = \{(t, t^2, t^3); t \in \mathbb{C}\} \subset \mathbb{A}^3(\mathbb{C})$. Show that C is an irreducible closed subset of $\mathbb{A}^3(\mathbb{C})$. Find generators for the ideal $I(C)$.

2. Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3(\mathbb{C})$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

Solution 2.

1. We have a continuous map surjecting on the twisted cubic curve.

   $$\begin{aligned} f \ : \ \mathbb{A}^1(\mathbb{C}) &\longrightarrow \mathbb{A}^3(\mathbb{C}) \\ t &\mapsto (t, t^2, t^3) \end{aligned}$$

   If $C = F_1 \cup F_2$, with F_1 and F_2 Zariski closed subsets, $\mathbb{A}^1 = f^{-1}(F_1) \cup f^{-1}(F_2)$. $\mathbb{A}^1(\mathbb{C})$ is Zariski-irreducible so without lost of generality $f^{-1}(F_1) = \mathbb{A}^1(\mathbb{C})$ but then $C = F_1$. Hence C is irreducible. Moreover C is defined by the ideal

   $$I(C) = (Z - X^3, Y - X^2)$$

   so it is closed.

2. We can decompose

   $$V(X^2 - YZ, XZ - X) = V(X^2 - YZ, X(Z - 1))$$

   $$V = V_1 \cup L_2 \cup L_3$$

   where $V_1 = V(Z - 1, X^2 - Y)$ is an irreducible curve, $L_2 = V(X, Y)$ and $L_3 = V(X, Z)$ are lines.

Exercise 3.3. For topological spaces X and Y, the opens of the product topology on $X \times Y$ are *unions* of products of opens $U \times V$, where $U \subseteq X$ and $V \subseteq Y$. A topological space X is called *Hausdorff* if for any pair of points $x_1 \neq x_2 \in X$, there exist open subsets $U, V \subseteq X$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. A topological space G with an abstract group structure is called a *topological group* if the multiplication and inverse laws are continuous. Let $n \geq 1$.

1. Is the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$ (each copy of $\mathbb{A}^1_k$ being endowed with the Zariski topology) the same as the Zariski topology on $\mathbb{A}^2_k$?

2. Is the Zariski topology on $\mathbb{A}^n_k$ Hausdorff?

3. Is $(\mathbb{A}^n_k, +)$ a topological group for the Zariski topology (assuming $\mathbb{A}^n_k \times \mathbb{A}^n_k \simeq \mathbb{A}^{2n}_k$ is endowed with the Zariski topology)?

Solution 3.

1. No. For example, $V(1 - XY)$ is a Zariski closed subset in $\mathbb{A}^2_k$ but it is not closed in the product topology (it is not a countable union of points or of products of points with $\mathbb{A}^1$).

2. No. Since any two open sets are dense in $\mathbb{A}^n_k$, their intersection cannot be empty.

3. Yes. Since $(x,y) \to x+y$ and $x \to -x$ are algebraic (i.e. given by polynomials), it is continous for the Zariski topology.

Exercise 3.4.

1. Show that any (correction : non empty) open subset of an irreducible topological space is irreducible and dense.
2. Show that the closure of an irreducible subset of a topological space is irreducible.

Solution 4. Recall that in general, S topological set irreducible if for all W_1, W_2 closed in S, $S = W_1 \cup W_2 \Rightarrow S = W_1$ or $S = W_2$.

1. Let $U \subset F$ open in F irreducible. Then $F = F \setminus \overline{U} \cup \overline{U}$ so $F = F \setminus \overline{U}$ or $F = \overline{U}$, thus $U = \emptyset$ or U is dense in F.

   Let W_1, W_2 closed in U such that $U = W_1 \cup W_2$ and $\overline{U}$ irreducible. Taking closure, $\overline{U} = \overline{W_1} \cup \overline{W_2}$. Without lost of generality, $\overline{U} = \overline{W_1}$. Now $U \setminus W_1$ is open in $\overline{U}$, so it is dense or empty. If it is dense, then $W \subset \overline{W} = \overline{U \setminus W}$ so that $W \subset \overline{U} \setminus U$, contradiction. So $U \setminus W_1$ is empty and $U = W_1$.

2. Similar argument.

Exercise 3.5. Let V an affine variety. Show that algebraic subsets of V are in one-to-one correspondence with radical ideals of $\Gamma(V)$. Show that under this correspondence, affine subvarieties correspond to prime ideals and points correspond to maximal ideals.

Solution 5. We have two maps

$$\{\text{Algebraic sets}\} \longrightarrow \{\text{Radical ideals}\}$$

$$V \mapsto I(V)$$

$$V(I) \leftarrow I$$

$I(V)$ is radical, the map is well-defined. Let's verify that both maps are inverse of each other, i.e. $I(V(I)) = I$. $I \subset I(V(I))$ is clear. $I(V(I)) = \mathsf{Rad}(I)$ is the content of Hilbert Nullstellensatz. As I is radical, $I(V(I)) = I$.

Suppose I is prime. Then if $V(I) = V_1 \cup V_2$ algebraic sets, then $V(I) = V(I(V_1)) \cup V(I(V_2))$ so $I(V_1)I(V_2) \subset I$. I is prime so without lost of generality $I(V_1) = I$ and $V(I) = V_1$. In turn, $V(I)$ is irreducible.

Suppose I is maximal. As $I \subset J$ implies $V(J) \subset V(I)$, I maximal implies that $V(I)$ does not have proper algebraic subset, so it is a point.

Arguments for converses statements are similar.